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Contact Problem of an Elastically Dissimilar Symmetric Wedge with a Half-Plane Including Convective Effect

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Abstract

In this paper, the contact problem between a dissimilar wedge and an elastic half-plane was studied. The investigated problem considered the convective effect which resembles a non-Hertzian contact problem. Unlike the previous studies, which focused to find the solutions of the singular integral equations, here, the contact problem was formulated based on the Muskhelishvili complex potential method for the two-dimensional elasticity. The problem formulation was converted into a Riemann-Hilbert problem. Finally, a closed-form solution for the Riemann-Hilbert problem in terms of the Muskhelishvili complex potential was derived. According to the results, the convective effect increased the contact half-length by a factor of 35% for the case of maximum material dissimilarity. Moreover, the complex potential for the condition of the similar materials was extracted in accordance with the previous studies.

***Keywords**:* Contact problem, Dissimilar materials, Convective effect, and Complex potential.

1. INTRODUCTION

In the original hypothesis of Hertz's theory the possibility of the convective effect was ignored, i.e., the surface particles that come into account are not the ones that were originally opposite to each other in the original configuration. In dissimilar contact problems, the convective effect causes the contacting bodies to experience different lateral displacements as opposed to the similar materials which have the same amount of lateral displacements (and therefore no convective effect).

The contact problem of dissimilar materials has been studied by several investigators due to its practical applications [1-3]. Hills and Sackfield [1] studied the contact problem of dissimilar elastic cylinders and derived expressions for the interior stress field. Later, sliding contact between a dissimilar wedge and a half plane was investigated by Truman et al., [2]. Truman and his colleagues considered full slip contact problem, and via Coulomb friction law, derived a closed-form solution for the pressure distribution in the terms of hypergeometric functions. Truman and Sackfield [3] obtained closed-form solutions for the stress field that is induced by wedge shape indenters for several cases. They investigated the normal and the sliding contact of a blunt wedge with a half-plane. Moreover, using Bertrand-Poincare lemma, they derived the closed-form solution for the fully coupled contact problem of dissimilar materials in terms of hypergeometric functions. The contact problem of the dissimilar wedge with a half-plane including convective effect was considered by Ciavarella and Hills [4]. They used an iterative scheme to solve the integral equations and obtained the pressure distributions for the symmetric wedge as well as for a parabolic punch.

The contact problem for an asymmetric wedge was addressed by Shahani et al. [5]. They utilized the Legendre polynomials to determine the contact pressure as well as the subsurface stress field. Ostryk and Shchokotova [6] applied the Wiener–Hopf method on the contact problem of a stamp with a rectilinear base and an elastic wedge. They proposed an analytical solution for the stresses distribution and the elastic deformation of the wedge surface. Khaleghian et al. [7] conducted a photoelasticity experimental method to extract the stress intensity factor for an edge crack loaded by an asymmetric tilted wedge. Also, they provided the finite element analysis for the mentioned crack-contact problem. Tan et al. [8] employed the digital image correlation (DIC) method to study the contact problem between a blunt wedge indenter and a soft material block. The deformation contours and the surface stresses are the main objectives of their study.

To the best of authors' knowledge of the stress field for the contact problem of the symmetric wedge with a half-plane including convective effect has not been investigated in the open literature. Thus, the main objective of this study was to formulate the problem using complex variables method. First, the governing integral equation associated with the convective effect problem was obtained. Then, the corresponding Riemann-Hilbert problem that is a boundary value problem of the third type with constant coefficients was derived using the concept of Muskhelishvili complex potential function. Finally, the Riemann-Hilbert problem was solved in closed form and Muskhelishvili complex potential was obtained. Additionally, the

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analytical solution for the interior stress field as well as the surface contact pressure was extracted. Moreover, the effect of material dissimilarity parameter on the pressure distribution and the normalized VonMises' parameter was investigated.

2. FORMULATION OF THE PROBLEM

2.1 2D CONTACT PROBLEMS WITH MIXED BOUNDARY CONDITIONS

Generally, 2D contact problems are categorized into three groups according to their boundary conditions. The most common types are the mixed boundary value problems where displacements are specified over the line of contact and surface tractions are specified out of contact zone (see Fig. 1).

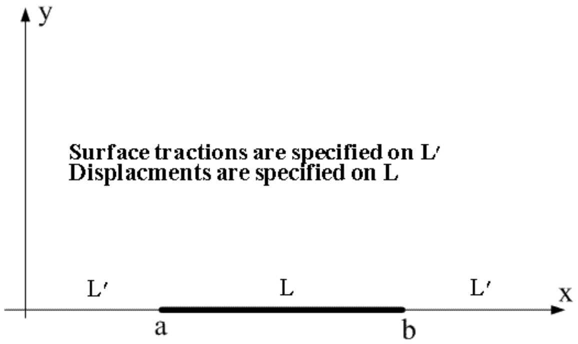

Fig. 1. Boundary condition of the mixed type contact problems.

According to the complex variables method of Muskhelishvili, a solution to the two-dimensional elasticity problems can be written [9]:

$$2\mu(u+iv)=\kappa\varphi(z)-z\overline{\varphi'}(\bar{z})-\overline{\psi}(\bar{z}) \qquad (1)$$

where $z=x+iy$ is the complex variable and (u, v) are the displacement components in x and y directions, and i = sqrt (-1) respectively. Furthermore, $\varphi(z)$ and $\psi(z)$ are two arbitrary harmonic functions. However, in problems that the contacting bodies are considered as half-plane and only occupy upper or lower half space, it can be shown that only one analytic function is required to do determine the elastic field [10]. Thus:

$$\sigma_{xx}+\sigma_{yy}=2[\phi(z)+\overline{\phi}(\bar{z})] \qquad (2)$$

$$\sigma_{yy}-i\tau_{xy}=\phi(z)-\phi(\bar{z})+(z-\bar{z})\overline{\phi'}(\bar{z}) \qquad (3)$$

$$2\mu\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)=\kappa\phi(z)+\phi(\bar{z})-(z-\bar{z})\overline{\phi'}(\bar{z}) \qquad (4)$$

2.2 THE RIEMANN-HILBERT PROBLEM

The general form of the Riemann-Hilbert problem is given by the following formula [9]:

$$\phi^{+}(t)-M(t)\phi^{-}(t)=m(t) \quad \text{on L} \qquad (5)$$

where the functions M(t) and m(t) are known as the coefficient and the free term of the Riemann-Hilbert problem, respectively. In addition, L is the contour or the arc that the problem is defined on, see Fig. 2. The general solution of this problem is given by the following formula [9, 11]:

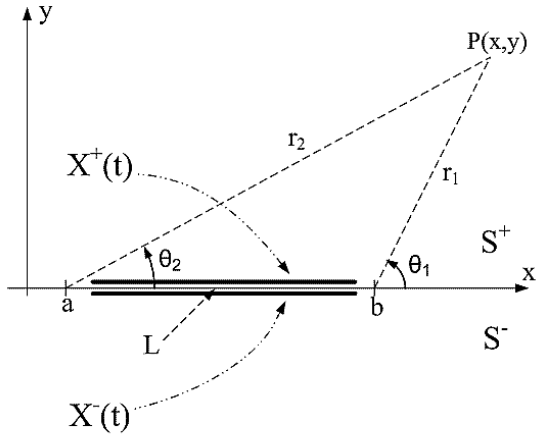

Fig. 2. Definition of parameters related to a Riemann-Hilbert problem defined over line L

$$\phi(z)=\frac{X(z)}{2\pi i}\int_{L}\frac{m(t)dt}{X^{+}(t)(t-z)}+X(z)P_{\xi-1}(z) \qquad (6)$$

where *X(z)* is the homogeneous solution of the Riemann-Hilbert problem and $X(t)^{+}$ is the value of *X(z)* when an arbitrary point *P* approaches to the *x*-axis from the upper half-plane, i.e., S^{+}. The function $P_{\xi-1}(z)$ is an arbitrary polynomial of the degree not greater than $\xi-1$, where ξ is known as the index of the solution. For the case $\xi<0$, it can be shown that if a bounded solution vanishing at infinity is required, the following condition should be satisfied [9].

$$\int_{L}\frac{t^{k}m(t)dt}{X^{+}(t)}=0,\ k=0,1,\ldots,-\xi-1 \qquad (7)$$

provided that the above condition is accomplished, the solution of the Eq.(5) is given by Eq.(6) with $P_{\xi-1}(z)=0$. The homogeneous solution, X(z), is given by the following:

$$\mathrm{X}(z)=(z-a)^{A_1+\lambda_1}(z-b)^{A_2+\lambda_2} \qquad (8)$$

where *a* and *b* are the ends of line *L* according to Fig. 2. Moreover, A_1 and A_2 are defined as follows:

$$\begin{cases} \mathrm{A}_1 = -\left.\dfrac{\log(M(t))}{2\pi i}\right|_{t=a} \\ \mathrm{A}_2 = \left.\dfrac{\log(M(t))}{2\pi i}\right|_{t=b} \end{cases} \tag{9}$$

The integer $\lambda_{n=1,\ 2}$ can be determined from the boundedness of the solution near the ends such that $\left|\lambda_n + \mathrm{A}_n\right| \le 1$. Finally, by substitution of Eq.(8) into Eq.(6) we arrive at:

$$\phi(z) = \frac{(z-a)^{\mathrm{A}_1+\lambda_1}(z-b)^{\mathrm{A}_2+\lambda_2}}{2\pi i e^{\pi i(\mathrm{A}_2+\lambda_2)}} \times \int_a^b \frac{m(t)dt}{(t-a)^{\mathrm{A}_1+\lambda_1}(b-t)^{\mathrm{A}_2+\lambda_2}(t-z)} \tag{10}$$

The general form of the Riemann-Hilbert problem and its solution was presented here. Further, it can be shown that the contact problem of the symmetric wedge with a half-plane including a convective effect takes a special form of the mentioned problem.

2.3 THE CONTACT PROBLEM OF THE SYMMETRIC WEDGE INCLUDING THE CONVECTIVE EFFECT

We considered the frictionless contact problem of a dissimilar symmetric wedge and a half-plane including the convective effect as shown in Fig. 3. The convective effect has the maximum influence at the absence of the friction because the action of the friction is always to resist the relative tangential motion. The basic equations that relate the surface tractions to the relative surface displacements in the frictionless contact problems are given as [10]:

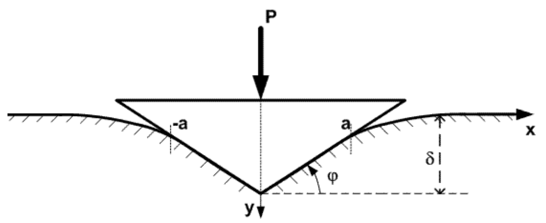

Fig. 3. Contact geometry of the symmetric wedge indenting the elastic half-plane

$$\frac{1}{A}\frac{\partial h(x)}{\partial x} = \frac{1}{\pi}\int_a^b \frac{p(\xi)d\xi}{x-\xi} \tag{11}$$

$$\frac{1}{A}\frac{\partial g(x)}{\partial x} = \beta p(x) \tag{12}$$

$$A = \frac{\kappa_1 + 1}{4\mu_1} + \frac{\kappa_2 + 1}{4\mu_2} \tag{13}$$

$$\beta = \frac{\Gamma(\kappa_1 - 1) - (\kappa_2 - 1)}{\Gamma(\kappa_1 + 1) + (\kappa_2 + 1)} \tag{14}$$

$$\Gamma = \frac{\mu_2}{\mu_1} \tag{15}$$

where the functions $h(x)$ and $g(x)$ are the surface relative normal and tangential displacements of contacting bodies respectively and $p(x)$ is the normal contact pressure. Furthermore, the constant A is the composite compliance and the constant β, which is referred to as Dundurs' constant, is a measure of elastic mismatch of the materials. The parameter κ_i is Kolosov's constant which is given by $\kappa_i = 3 - 4\upsilon_i$ under the plain strain conditions. The constant υ_i and μ_i are the Poisson's ratio and the shear modulus of the contacting bodies, respectively. It can be seen that equations (11) and (12) are coupled through constant β when the contacting bodies are dissimilar, i.e., $\beta \neq 0$. In some cases it is assumed that one body is more rigid than the other one, therefore, it can be shown that:

$$A = 2\frac{1-\upsilon^2}{E} \tag{16}$$

$$\beta = \frac{1-2\upsilon}{2(1-\upsilon)} \tag{17}$$

As mentioned earlier, when the convective effect is considered, the tangential displacements are taken into consideration in the derivation of the contact pressure relation. Thus, it is sufficient to replace the argument of the function $h(x)$ by the following relation [4]:

$$h(x) \rightarrow h(x - sgn(x)g(x)) \tag{18}$$

Substituting Eq.(18) into Eq.(11) and using the chain rule, equation (11) takes the following form:

$$\frac{1}{A}h'(x - sgn(x)g(x))(1 - sgn(x)\beta A p(x)) = \frac{1}{\pi}\int_a^b \frac{p(\xi)d\xi}{x-\xi} \tag{19}$$

Inasmuch as the derivative of the wedge profile is constant and the researchers considered the first order approximation, hence one may write:

$$h'(x - sgn(x)g(x)) \approx h'(x) \tag{20}$$

Now, the appropriate Riemann-Hilbert problem for the wedge contact problem can be constructed. According to the contact geometry of the symmetric wedge, shown in Fig. 3, the boundary conditions are given as follows:

$$\tau_{xy}(x,0)=q(x)=0 \quad \text{(on the entire } x \text{ axis)} \tag{21}$$

$$\sigma_{yy}(x,0)=p(x)=0 \quad \text{(outside the contact zone)} \tag{22}$$

$$\begin{aligned}\frac{\partial v}{\partial x}&=\frac{d}{dx}h(x-sgn(x)g(x))\\&=h'(x)(1-sgn(x)\beta Ap(x))\end{aligned} \tag{23}$$

(inside the contact zone)

$$\frac{\partial u}{\partial x}=g'(x)=\beta Ap(x) \tag{24}$$

(inside the contact zone)

Referring to Fig. 3 for the geometry of the symmetric wedge:

$$h'(x)=-\varphi\,\text{sgn}(x)=\begin{cases}\varphi & -a\prec x\prec 0\\ -\varphi & 0\prec x\prec a\end{cases} \tag{25}$$

By elimination of the stress σ_{yy} between Eq.(3) and its complex conjugate and application of Eq.(21) on the result, one may arrive at:

$$\phi^{+}(x)+\overline{\phi}^{+}(x)=\phi^{-}(x)+\overline{\phi}^{-}(x) \tag{26}$$

Thus, the function $\phi(z)+\overline{\phi}(z)$ is analytic in the entire plane. Further, it vanishes at the infinity, so it is zero everywhere [9], thus:

$$\phi(z)=-\overline{\phi}(z) \tag{27}$$

Similarly, the term $\frac{\partial u}{\partial x}$ can be eliminated between Eq.(4) and its complex conjugate and by applying Eq.(27) on the result, one may arrive at:

$$4\mu i\frac{\partial v}{\partial x}=(1+\kappa)\phi^{+}(x)+(1+\kappa)\phi^{-}(x) \tag{28}$$

According to the Plemelj formulas [9- 11], it can be shown that:

$$\phi^{+}(x)-\phi^{-}(x)=p(x) \tag{29}$$

By substituting Eq.(29) into Eq.(23) and the result into Eq.(28), after some simplifications the following relation is obtained:

$$\begin{aligned}&\frac{-4\mu i\,\varphi sgn(x)}{1+\kappa-4\beta A\mu i\varphi}\\&=\phi^{+}(x)+\frac{1+\kappa+4\beta A\mu i\varphi}{1+\kappa-4\beta A\mu i\varphi}\phi^{-}(x)\end{aligned} \tag{30}$$

Comparing Eq.(5) and Eq.(30) shows that Eq.(30) is a special form of the Riemann-Hilbert problem where the function *M(t)* is a constant.

3. SOLUTION

The Riemann-Hilbert problem (30) can be solved using the procedure described in section (2.2). Since the problem investigated is an incomplete type of a contact problem, the solution should be bounded at both ends. Thus, by using Eq.(9) and Eq.(30) we have:

$$\text{A}_1=-\text{A}_2=-\left(\frac{1}{2}+\frac{1}{\pi}\tan^{-1}\left(\frac{4\beta A\mu\varphi}{1+\kappa}\right)\right) \tag{31}$$

$$\lambda_1=1,\lambda_2=0$$

Finally, using Eq.(10) and noting that the wedge profile is symmetric, one may arrive at:

$$\begin{aligned}\phi(z)&=\frac{2\mu\varphi i\cos(\pi j)(z+a)^{1-\gamma}(z-a)^{\gamma}}{\pi(1+\kappa)}\\&\int_{-a}^{a}\frac{sgn(t)dt}{(a-t)^{\gamma}(t+a)^{1-\gamma}(t-z)}\end{aligned} \tag{32}$$

where:

$$\begin{cases}\gamma=\dfrac{1}{2}+\dfrac{1}{\pi}\tan^{-1}\left(\dfrac{4\beta A\mu\varphi}{1+\kappa}\right)\\ j=\dfrac{1}{\pi}\tan^{-1}\left(\dfrac{4\beta A\mu\varphi}{1+\kappa}\right)\end{cases} \tag{33}$$

In order to find $\phi(z)$ in Eq. 32, the following integral must be evaluated:

$$I_m=\int_{-a}^{a}\frac{sgn(t)dt}{(a-t)^{\gamma}(t+a)^{1-\gamma}(t-z)} \tag{34}$$

The following normalization is used to normalize the limits of the integral:

$$\begin{cases}t=as\\ z=aw\end{cases}\rightarrow I_m=\frac{1}{a}\int_{-1}^{1}\frac{sgn(s)ds}{(1-s)^{\gamma}(s+1)^{1-\gamma}(s-w)} \tag{35}$$

The integral in Eq. (35) can be written in the following form:

$$\begin{aligned}I_m=&\frac{1}{a}\{\int_{-1}^{1}\frac{ds}{(1-s)^{\gamma}(s+1)^{1-\gamma}(s-w)}\\&-2\int_{-1}^{0}\frac{ds}{(1-s)^{\gamma}(s+1)^{1-\gamma}(s-w)}\}\end{aligned} \tag{36}$$

Assuming that $\mathrm{Re}(z) > 0$, the first integral in the curly bracket of Eq.(36) can be evaluated by means of the residue method in the complex theory of functions [12]. Thus:

$$I_1 = \int_{-1}^{1} \frac{ds}{(1-s)^{\gamma}(s+1)^{1-\gamma}(s-w)} = \frac{-\pi}{\sin(\pi\gamma)(w-1)^{\gamma}(w+1)^{1-\gamma}} \quad (37)$$

On the other hand, using the following transformation into the second integral of Eq.(36) and referring [9], yields:

$$s = \frac{1-u}{1+u} \quad (38)$$

$$I_2 = -\int_{-1}^{0} \frac{ds}{(1-s)^{\gamma}(s+1)^{1-\gamma}(s-w)} = \frac{1}{1-w}\int_{0}^{1} \frac{u^{-\gamma}du}{1-\frac{1+w}{1-w}u} = \frac{1}{1+w}B(\gamma,1)_2 F_1(1,\gamma;1+\gamma;\frac{1-w}{1+w}) \quad (39)$$

where ${}_2F_1$ is the hypergeometric function with one variable and $B(\eta,\theta)$ is the Beta function. Making substitution of appropriate values into Eq.(32) yields:

$$\phi(z) = \frac{2\mu\varphi i\cos(\pi j)}{\pi(1+\kappa)} \left\{ \frac{-\pi}{\sin(\pi\gamma)} + 2\left(\frac{z-a}{z+a}\right)^{\gamma} B(\gamma,1)\,{}_2F_1(1,\gamma;1+\gamma;\frac{a-z}{a+z}) \right\}, \quad \mathrm{Re}(z) > 0 \quad (40)$$

By a similar approach, it can be shown that if $\mathrm{Re}(z) < 0$ then the solution of Eq.(32) takes the following form:

$$\phi(z) = \frac{2\mu\varphi i\cos(\pi j)}{\pi(1+\kappa)} \left\{ \frac{\pi}{\sin(\pi\gamma)} - 2\left(\frac{z+a}{z-a}\right)^{1-\gamma} B(1-\gamma,1)_2 F_1(1,1-\gamma;2-\gamma;\frac{a+z}{a-z}) \right\}, \quad \mathrm{Re}(z) < 0 \quad (41)$$

As mentioned earlier, the problem of the form of Eq.(30) has a bounded solution provided that the consistency condition, Eq.(7), is satisfied. Index of the solution is $\xi = -1$, thus the following condition must be fulfilled:

$$\int_{L} \frac{m(t)dt}{X^{+}(t)} = 0 \quad \Rightarrow \quad \int_{-a}^{a} \frac{sgn(t)dt}{(a-t)^{\gamma}(a+t)^{1-\gamma}} = 0 \quad (42)$$

Using the transformation $t = as$ in Eq.(42) results in:

$$I = \int_{0}^{1} (1-s)^{-\gamma}(1+s)^{\gamma-1}ds - \int_{-1}^{0} (1-s)^{-\gamma}(1+s)^{\gamma-1}ds \quad (43)$$

Moreover, changing $s \to -s$ in the second integral of Eq.(43) leads to:

$$I = \int_{0}^{1} (1-s)^{-\gamma}(1+s)^{\gamma-1}ds - \int_{0}^{1} (1+s)^{-\gamma}(1-s)^{\gamma-1}ds \quad (44)$$

Referring [9], Eq.(44) becomes

$$I = \frac{1}{1-\gamma}\,{}_2F_1(1,1-\gamma;2-\gamma;-1) - \frac{1}{\gamma}\,{}_2F_1(1,\gamma;1+\gamma;-1) \quad (45)$$

After some simplification, one may arrive at the following solution form:

$$I = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n}{n^2-\gamma^2} - \frac{1}{\gamma} \quad (46)$$

The series in Eq.(46) is an alternating and conditionally convergent series but the numerical study shows that the solution of Eq.(46) does not tend to zero exactly. This error arises due to the approximation used in Eq.(20) since the argument of sign function replaced with *x*. This substitution causes the symmetry of the integration interval about zero. Also, the mentioned assumption leads to minor differences between the complex potential derived in Eqs.(40) and (41). Hence, it causes some asymmetries in the solution with respect to the *y*-axis. Since the geometry of the problem is symmetric with respect to the *y*-axis, the researchers considered the solution in the region of $\mathrm{Re}(z) > 0$ for the entirety of the plane.

The correctness of the complex potential given by Eqs.(40) and (41) can be investigated readily. First, it should be pointed out that if we let $\beta = 0$, we will achieve normal contact problem of symmetric wedge and a half-plane with similar materials. Now, consider the complex potential given by Eq. (40) and put $\beta = 0$, thus we have:

$$\beta = 0 \rightarrow \begin{cases} \gamma = \dfrac{1}{2} \\ j = 0 \end{cases} \quad (47)$$

$$\frac{2\pi i\,(1+\kappa)}{4\mu}\phi(z) = -\varphi\left\{-\pi + 2\left(\frac{z-a}{z+a}\right)^{\frac{1}{2}} B\left(\frac{1}{2},1\right)\, {}_2F_1\left(1,\frac{1}{2};\frac{3}{2};\frac{a-z}{a+z}\right)\right\} \quad (48)$$

After expanding the hypergeometric function and some manipulations, one may arrive at:

$$\frac{2\pi i\,(1+\kappa)}{4\mu}\phi(z) = -\varphi\left\{-\pi + 4\left[\left(\frac{z-a}{z+a}\right)^{\frac{1}{2}} - \frac{1}{3}\left(\frac{z-a}{z+a}\right)^{\frac{3}{2}} + \cdots\right]\right\} \quad (49)$$

On the other hand, the Maclaurin's series of the inverse tangent function is given by the following formula:

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad (50)$$

Using the series expansion given by Eq.(50), the series in Eq.(49) can be written in a more concise form as follows:

$$\frac{2\pi i\,(1+\kappa)}{4\mu}\phi(z) = -\varphi\left\{-\pi + 4\tan^{-1}\left(\sqrt{\frac{z-a}{z+a}}\right)\right\} \quad (51)$$

Noting that:

$$\tan^{-1}\left(\sqrt{\frac{1-t}{1+t}}\right) = \frac{\pi}{2} - \frac{1}{2}\cos^{-1}(-t) \quad (52)$$

and by letting $t = a/z$ in Eq.(51) and using Eq.(52), one arrives at:

$$\frac{2\pi i\,(1+\kappa)}{4\mu}\phi(z) = -\varphi\left\{\pi - 2\cos^{-1}\left(-\frac{a}{z}\right)\right\} \quad (53)$$

Moreover, we have:

$$\cos^{-1} t = \frac{\pi}{2} - \sin^{-1} t \quad (54)$$

thus:

$$\frac{2\pi i\,(1+\kappa)}{4\mu}\phi(z) = 2\varphi\sin^{-1}\left(\frac{a}{z}\right) \quad (55)$$

The presented result in Eq.(55) corresponds with the solutions for the symmetric wedge problem given in Refs. [3, 14]. The same result will be achieved if the complex potential in Eq.(41) is selected beforehand.

Having the complex potential at hand, the pressure distribution over the contact surface can be obtained using the Plemelj formula, Eq.(29). Referring to Fig. 2, the branch cut values of the complex potential in Eq.(40) can be determined readily, hence:

$$p(x) = \frac{-8\mu\varphi\cos^2(\pi j)}{\pi(1+\kappa)\gamma}\left\{\left(\frac{a-x}{a+x}\right)^{\gamma} {}_2F_1\left(1,\gamma;1+\gamma;\frac{a-x}{a+x}\right)\right\} \quad (56)$$

As the pressure distribution on the contacting surface was determined, the contact area length can be obtained by means of the vertical equilibrium condition as follows:

$$P = -2\int_0^a p(x)dx = \frac{16\mu\varphi\cos^2(\pi j)}{\pi(1+\kappa)\gamma}\int_0^a\left\{\left(\frac{a-x}{a+x}\right)^{\gamma} {}_2F_1\left(1,\gamma;1+\gamma;\frac{a-x}{a+x}\right)\right\}dx \quad (57)$$

Defining an appropriate transformation renders the integral in Eq.(57) tractable:

$$t = \frac{a-x}{a+x} \quad (58)$$

$$\frac{\pi A\gamma P}{8\varphi\cos^2(\pi j)} = a\int_0^1 t^{\gamma}(1+t)^{-2}\, {}_2F_1(1,\gamma;1+\gamma;t)dt \quad (59)$$

Finally, by evaluating the integral in Eq. (59) by referring [13], one may arrive at:

$$\frac{a}{a_0} = \frac{\pi\gamma}{\cos^2(\pi j)\, {}_3F_2\left(1,2,1;1+\gamma,2;\frac{1}{2}\right)} \quad (60)$$

where $a_0 = (PA/2\varphi)$ is the contact area length of the symmetric wedge problem with similar materials. As a general rule [15], a generalized hypergeometric series ${}_pF_q$, converges for $|z|<1$ when $p = q+1$. A numerical study shows that the ratio of a/a_0

converges to unity quite rapidly as $\gamma \to 1/2$, i.e., the case of similar materials.

4. RESULTS

Using the expression derived in the previous section, the effect of convection on the pressure distribution and the contact area size can be studied. In practice, the value of $|\beta|$ rarely exceeds 0.4 but the researchers highlighted the convection effect by taking $\beta = 0.5$, i.e., the maximum elastic mismatch. The present case arose when the indenter was considered rigid. The pressure distribution for the symmetric wedge including and excluding the convective effect is shown in Fig. 4. The results showed that the behavior of the pressure distribution is consistent with Ref. [4] but the contact area length increases up to 35% in the present study. On the other hand, behavior of the pressure distribution near the wedge apex is smoother for the case including the convective effect with respect to the case excluding it. Although pressure distribution is calm near the apex, it reaches to infinity unlike the result of Ref. [4]. Moreover, the behavior of the pressure distribution due to the change of the material dissimilarity parameter, β, has been studied in Fig. 5. It can be seen that the rise of the material dissimilarity causes the contact area length increases while the sharpness of the pressure profile decreases near the wedge apex. It may be interpreted that due to the lateral displacements imposed by the convective effect, the particles within the contacting bodies reach to their equilibrium state in a farther distance with respect to the case which excludes the convective effect.

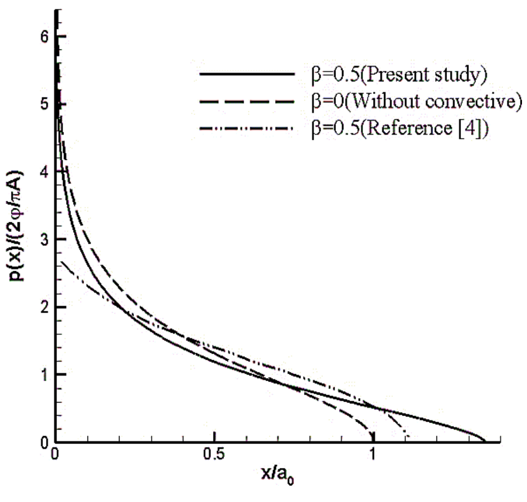

Fig. 4. Pressure distribution for symmetric wedge including convective effect and excluding it ($\varphi = \pi/5$)

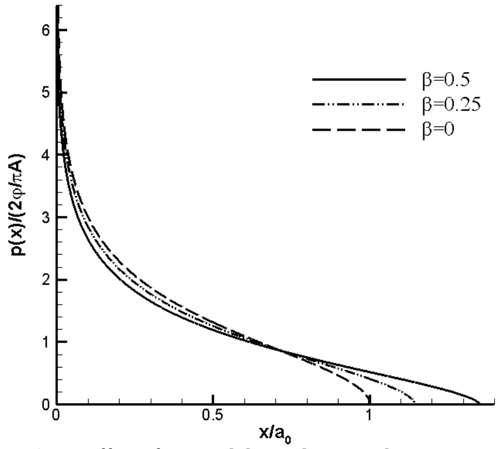

Fig. 5. Effect of material dissimilarity on the pressure distribution and contact area size ($\varphi = \pi/5$)

Design purposes required the state of stress to be determined everywhere in the half-plane to identify the critical stress zones. Here, we are interested in the investigation of the convective effect influences on the stress state which is induced by the wedge indentation. A common method for the investigation of the stress state is the contours of normalized Von Mises' parameter (NVMP) which is given by the following:

$$NVMP = \frac{\pi A\sqrt{J_2}}{2\varphi} \tag{61}$$

where J_2 is the second principal invariant of the stress deviator tensor which plays an important role in the mathematical theory of plasticity

The contour plot of NVMP is shown in Fig. 6 under the plane strain conditions for the symmetric wedge including and excluding the convective effect. It can be seen that the critical stresses occur at the wedge apex for both cases but the difference between the contours become larger when the convective effect is off. In addition, for the case without the convective effect, the values of NVMP around the wedge apex are approximately 20% greater than the case includes the convective effect. As the contours of NVMP are contracted when the convective effect is included, the plastic zone area, in this case, is smaller. Thus, imposing the effect of convection enables us to obtain an appropriate estimation of the plastic zone size within the body. It is worthy to mention that the plastic zone size plays an important role in damage plasticity and fracture mechanics analyses. Therefore, a better calculation of plastic zone size is rewarding in order to avoid overestimation of the design process. Concisely, an analytical model is more realistic if the convective effect is included.

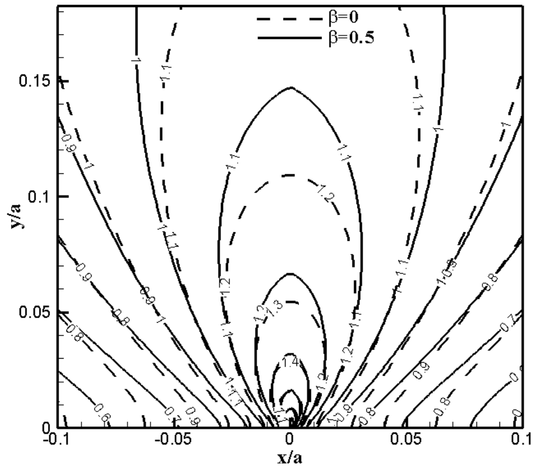

Fig. 6. Contours of NVMP including convective effect and excluding it (plane strain condition, $\nu = 0.3$)

5. CONCLUSIONS

According to the complex variables method in elasticity, a 2D contact problem has an associated boundary value problem which is a special form of the Riemann-Hilbert problem. The solution of the Riemann-Hilbert problem delivered the complex potential for the contact problem. In the present study, the associated Riemann-Hilbert problem for the contact problem of the symmetric dissimilar wedge including convective effect was developed and the Muskhelishvili complex potential was derived in the closed-form. The effect of convection on the pressure distribution as well as NVMP was investigated. Results showed that the material dissimilarity causes the rise of the contact length up to 35% in the case of $\beta = 0.5$. In addition, consideration of the convective effect leads to the shrinkage of NVMP contours, the decrease of plastic zone area and the reduction of states with severe stresses.

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